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Some remarks on the integration of the Poisson algebra

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Abstract

We prove the integrability of the Poisson algebra of functions with compact supports of a noncompact manifold. We also determine a Lie subalgebra of vector fields which, weakly, integrate the Poisson algebra of a not necessarily compact manifold covered by an exact symplectic manifold.

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1. Introduction

Lie's third theorem asserts that any finite dimensional Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}$ of some Lie group G. Recall that $\mathfrak{g}$ consists of left invariant vector fields on G.

The assertion of Lie's third theorem is wrong in general for infinite dimensional Lie algebras. For counter-examples, we refer to [9,14].

An (infinite dimensional) topological Lie algebra l is said to be *integrable* if there exists an (infinite dimensional) Lie group G which is modeled on a Lie algebra isomorphic to l. In such a case l will be said to be *integrated* by G.

One of the main examples of infinite dimensional Lie algebras is the Lie algebra Vect(M) of vector fields on a compact manifold M which is integrated by the group Diff(M) of all diffeomorphisms of M [10,14–16], both spaces being endowed with the C^∞ topology.

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An idea to integrate a Lie algebra l would be first to find a Lie algebra isomorphism between l and a Lie subalgebra $\mathfrak{g}$ of Vect(M), for some smooth manifold M; we call this step *the weak integration*, and then check whether $\mathfrak{g}$ is the Lie algebra of some Lie subgroup of Diff(M). This brings us to the question whether the second Lie's fundamental theorem is true in the infinite dimensional setting. Unfortunately, there are Lie subalgebras of Vect(M) which may not be integrated by any subgroup of Diff(M) or by any other group. See for instance [15,16]. We refer to [11] where sufficient conditions for the integration of Lie subalgebras of Vect(M) are given.

This note focuses on the Lie algebra $P(M, \Omega)$ of smooth real valued functions on a symplectic manifold (M, Ω), with the Poisson bracket. This Lie algebra is called the Poisson algebra of (M, Ω).

It is known that if the symplectic manifold M is compact, then $P(M, \Omega)$ is integrable (see for instance [1]). Here we provide a new proof which has the advantage to generalize to the noncompact case and establishes the integrability of the Lie algebra $P_c(M, \Omega)$ of smooth functions with compact supports on a noncompact symplectic manifold (M, Ω) (Theorem 7).

In contrast with the functional analysis approaches of the authors cited above, our methods are more geometric in nature and use essentially various restrictions of the Weinstein chart for symplectic diffeomorphisms [20].

Michor [13] has introduced a topology on the group Diff(M) of smooth diffeomorphisms of a noncompact manifold, which makes it a Lie group modeled on Vect(M). But we do not know of any theorem dealing with the integrability of Lie subalgebras of Vect(M), with unrestricted supports.

It is well known that if Ω is exact, then $P(M, \Omega)$ is isomorphic to the Lie algebra $\mathcal{L}_\omega(M \times S^1)$ of vector fields whose local flow preserve the *contactisation* ω of the exact symplectic form Ω. We generalize this to symplectic manifolds (M, Ω) such that the pullback of Ω to the universal covering $\widehat{M}$ of M is exact. We identify the subalgebra of $\mathcal{L}_\omega(\widehat{M} \times S^1)$ which is isomorphic to $P(M, \Omega)$ (Theorem 3).

Finally we show that the group of strictly contact diffeomorphisms of a compact regular contact manifold is modeled on the Lie algebra of strictly contact vector fields (Theorem 9).

From now on all manifolds will be assumed to be connected.

2. Weak integration

If (M, Ω) is a symplectic manifold, the space $C^\infty(M)$ of real valued functions on M is equipped with the *Poisson bracket*. Every $f \in C^\infty(M)$ defines a vector field X_f given by the equation: $i_{X_f}\Omega = \mathrm{d}f$. The Poisson bracket of two functions is then defined by

$$\{f, g\} = \Omega(X_f, X_g).$$

We point out that $[X_f, X_g] = X_{\{f,g\}}$. $(C^\infty(M), \{\,,\})$ is an infinite dimensional Lie algebra which we denote by $P(M, \Omega)$, it is called the Poisson algebra of (M, Ω); it plays a major role in several parts of Mathematics and Mechanics. Similary $P_c(M, \Omega)$ denotes the Poisson algebra of compactly supported functions.

Proposition 1. *If M is compact, then $P(M, \Omega)$ is weakly integrable. For noncompact M, then the Poisson algebra $P_c(M, \Omega)$ of compactly supported functions is weakly integrable.*

Proof. Let $\mathcal{L}_\Omega(M)$ be the Lie algebra of symplectic vector fields, i.e. vector fields X on M such that $i(X)\Omega$ is closed and let $\mathrm{Ham}(M)$ be the ideal of $\mathcal{L}_\Omega(M)$ consisting of vector fields X such that $i(X)\Omega$ is exact. The symplectic gradient mapping $f \mapsto X_f$ is a surjective Lie algebra homomorphism from $P(M, \Omega)$ to $\mathrm{Ham}(M)$, with kernel $\mathbb{R}$, provided that M is connected. In other terms, we have an exact sequence of Lie algebras:

$$0 \to \mathbb{R} \to P(M, \Omega) \to \mathrm{Ham}(M) \to 0.$$

According to Dumortier–Takens [7], the exact sequence above splits if and only if M is compact. Hence if M is compact, $P(M, \Omega)$ is isomorphic (as a Lie algebra) with $\mathrm{Ham}(M) \times \mathbb{R}$, which is a Lie subalgebra of $\mathrm{Vect}(M \times \mathrm{S}^1)$. Hence $P(M, \Omega)$ is weakly integrable if M is compact.

In the noncompact case, the symplectic gradient map is an isomorphism between $P_c(M, \Omega)$ and $\mathrm{Ham}_c(M) \subset \mathrm{Vect}(M)$, the Lie algebra of compactly supported hamiltonian vector fields. □

Remark 2. If the symplectic form Ω of a (compact or noncompact) symplectic manifold M has discrete group of periods, then there exists an S^1 principal bundle over M: $\pi : E \to M$, with a contact form ω on E such that $\pi^*\Omega = \mathrm{d}\omega$. For each function $f \in P(M, \Omega)$, one defines a unique vector field Y_f on E characterized by

$$i(Y_f)\omega = f \circ \pi \quad \text{and} \quad i(Y_f)\,\mathrm{d}\omega = -\mathrm{d}(f \circ \pi).$$

One proves that $Y_f \in \mathcal{L}_\omega(E)$ and that $f \mapsto Y_f$ realizes a Lie algebra isomorphisms between $P(M, \Omega)$ and $\mathcal{L}_\omega(E)$ [8,18]. Here $\mathcal{L}_\omega(E)$ denotes the Lie algebra of vector fields X on E such that $L_X\omega = 0$, where L_X stands for the Lie derivative in the direction X. This shows that $P(M, \Omega)$ is weakly integrable if Ω has discrete periods.

In particular if Ω is exact, $E = M \times \mathrm{S}^1$ is trivial and $\omega = \pi_1^*\alpha + \pi_2^*(\mathrm{d}\theta)$, where π_i are the projections from $E \times \mathrm{S}^1$ to each factor, α is a primitive of Ω, i.e. $\Omega = \mathrm{d}\alpha$ and $\mathrm{d}\theta$ is the canonical 1-form on S^1.

Suppose now the pullback $\widehat{\Omega} = p^*\Omega$ of the symplectic form Ω on M to its universal covering $\widehat{M}$ is exact. Here $p : \widehat{M} \to M$ is the covering projection. The injective homomorphism

$$p^* : P(M, \Omega) \to P(\widehat{M}, \widehat{\Omega}), \qquad f \mapsto f \circ p$$

identifies $P(M, \Omega)$ with a subalgebra of $P(\widehat{M}, \widehat{\Omega}) \approx \mathcal{L}_\omega(\widehat{M} \times \mathrm{S}^1)$. Our next task is to determine exactly the subalgebra of $\mathcal{L}_\omega(\widehat{M} \times \mathrm{S}^1)$ with which $P(M, \Omega)$ is isomorphic.

Theorem 3. *Let (M, Ω) be a symplectic manifold such that the pullback of Ω to the universal covering of M is exact, then $P(M, \Omega)$ is isomorphic to the Lie algebra $\mathcal{L}$ of vector fields which strictly preserve the contact form of $\widehat{M} \times S^1$ and which are invariant by a strictly contact action of a central extension of the fundamental group of M.*

Proof. Recall we denoted by $p : \widehat{M} \to M$ the universal covering of M, and by $\widehat{\Omega} = p^*\Omega$ the pulled back symplectic form on $\widehat{M}$. The fundamental group $\pi_1(M)$ acts on $\widehat{M}$ by deck transformations. Moreover $\pi_1(M)$ acts as a subgroup of $\mathrm{Diff}_{\widehat{\Omega}}(\widehat{M})$ the group of symplectic diffeomorphisms of the covering; indeed we have

$$\forall c \in \pi_1(M), \quad c^*\widehat{\Omega} = c^*p^*\Omega = p^*\Omega = \widehat{\Omega}, \text{ since } p \circ c = p.$$

We suppose now that $\widehat{\Omega}$ is exact, and denote by α a potential: $\widehat{\Omega} = \mathrm{d}\alpha$.

Keeping the previous notations, let us consider $Y = \widehat{M} \times S^1$ on which we define the 1-form: $\omega = \pi_1^*\alpha + \pi_2^*(\mathrm{d}\theta)$. We will denote the contact form ω by

$$\omega = \alpha + \frac{\mathrm{d}z}{\mathrm{i}z}.$$

We will need the following:

Lemma 4. *The fundamental group $\pi_1(M)$ of M admits a real central extension which acts on Y as a subgroup of the group of strictly contact diffeomorphisms of (Y, ω).*

Proof. The relation $c^*\widehat{\Omega} = \widehat{\Omega}$, $c \in \pi_1(M)$ implies $\mathrm{d}(c^*\alpha - \alpha) = 0$. Since $\widehat{M}$ is simply connected, there exists a unique function $f_c \in C^\infty(\widehat{M})$, such that $c^*\alpha - \alpha = \mathrm{d}f_c$ and $f_c(x_0) = 0$ for a fixed base point $x_0 \in \widehat{M}$.

Given c_1 and c_2 in $\pi_1(M)$ we have

$$\begin{aligned}\mathrm{d}(f_{c_1c_2}) &= (c_1 \circ c_2)^*\alpha - \alpha = c_2^*(c_1^*\alpha - \alpha) + c_2^*\alpha - \alpha \\ &= c_2^*(\mathrm{d}f_{c_1}) + \mathrm{d}f_{c_2} = \mathrm{d}(f_{c_1} \circ c_2 + f_{c_2}).\end{aligned}$$

Therefore

$$f_{c_1c_2} = f_{c_1} \circ c_2 + f_{c_2} + u(c_1, c_2),$$

here $u(c_1, c_2)$ is constant. We claim that u is a 2-cocycle of $\pi_1(M)$ with real values. Indeed for any three elements in $\pi_1(M)$, we have

$$f_{c_1c_2c_3} = f_{c_1c_2} \circ c_3 + f_{c_3} + u(c_1c_2, c_3),$$

on the other hand we also may write

$$f_{c_1c_2c_3} = f_{c_1} \circ c_2c_3 + f_{c_2c_3} + u(c_1, c_2c_3)$$

developing $f_{c_1c_2}$ and $f_{c_2c_3}$ in each expression, leads after simplification to a 2-cocycle relation

$$u(c_1, c_2) + u(c_1c_2, c_3) = u(c_2, c_3) + u(c_1, c_2c_3).$$

We pointed out that this cocycle has generally nontrivial cohomology class. It determines a central extension of $\pi_1(M)$ by $\mathbb{R}$, i.e. a group structure on the cartesian product $\pi_1(M)\times\mathbb{R}$ with a modified product law

$$(c_1,r)(c_2,t)=(c_1c_2,r+t+u(c_1,c_2)).$$

We shall denote by $\pi_1(M)\rtimes\mathbb{R}$ this extension. For all $(c,r)\in\pi_1(M)\rtimes\mathbb{R}$ we define

$$(c,r):\widehat{M}\times S^1\to\widehat{M}\times S^1 \quad \text{by} \quad (c,r)(x,z)=(c(x),z\mathrm{e}^{-\mathrm{i}f_c(x)}\mathrm{e}^{\mathrm{i}r}).$$

We have now an action of the central extension on Y, indeed:

$$\begin{aligned}(c_1,r)\big[(c_2,t)(x;z)\big]&=(c_1,r)(c_2(x),z\mathrm{e}^{-\mathrm{i}f_{c_2}(x)}\mathrm{e}^{\mathrm{i}t})\\&=(c_1(c_2(x)),z\mathrm{e}^{-\mathrm{i}f_{c_1}(c_2(x))}\mathrm{e}^{-\mathrm{i}f_{c_2}(x)}\mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}r})\\&=(c_1\circ c_2(x),z\mathrm{e}^{-\mathrm{i}[f_{c_1}\circ c_2+f_{c_2}](x)}\mathrm{e}^{\mathrm{i}(t+r)})\\&=(c_1\circ c_2(x),z\mathrm{e}^{-\mathrm{i}f_{c_1c_2}(x)}\mathrm{e}^{\mathrm{i}(t+r+u(c_1,c_2))})\\&=(c_1c_2,r+t+u(c_1,c_2))(x,z)\\&=(c_1,r)(c_2,t)(x,z).\end{aligned}$$

It is clear that this action lifts the one of $\pi_1(M)$ on $\widehat{M}\times S^1$.

Let us now compute $(c,r)^*\omega$. We have

$$(c,r)^*\omega=(c,r)^*\left(\alpha+\frac{\mathrm{d}z}{\mathrm{i}z}\right)=\alpha+\mathrm{d}f_c+(c,r)^*\left(\frac{\mathrm{d}z}{\mathrm{i}z}\right),$$

but

$$(c,r)^*\,\mathrm{d}z=\mathrm{d}(z\mathrm{e}^{-\mathrm{i}f_c(x)}\mathrm{e}^{\mathrm{i}r})=(\mathrm{d}z-\mathrm{i}z\,\mathrm{d}f_c)\mathrm{e}^{-\mathrm{i}f_c(x)}\mathrm{e}^{\mathrm{i}r},$$

hence

$$(c,r)^*\frac{\mathrm{d}z}{\mathrm{i}z}=\frac{\mathrm{d}z}{\mathrm{i}z}-\mathrm{d}f_c,$$

finally

$$(c,r)^*\omega=\alpha+\mathrm{d}f_c+\frac{\mathrm{d}z}{\mathrm{i}z}-\mathrm{d}f_c=\omega.$$ □

Proof of Theorem 3 (continued). Consider the following subalgebra of Vect(Y):

$$\mathcal{L}=\big\{X\in\mathrm{Vect}(Y)/\forall(c,r)\in\pi_1(M)\rtimes\mathbb{R}\ \ (c,r)_*X=X \text{ and } L_X\omega=0\big\}.$$

Denote by $\sigma=p\circ\pi$ the natural projection from Y onto M. Let ξ be the Reeb field of ω. Recall that ξ is uniquely characterized by the following equations: $i(\xi)\omega=1$ and $i(\xi)\,\mathrm{d}\omega=0$.

The vector field ξ is vertical for the projection σ, i.e. $\sigma_*\xi=0$. Indeed

$$0=i(\xi)\,\mathrm{d}\omega=i(\xi)(\sigma^*\Omega)=\sigma^*(i(\sigma_*\xi)\Omega).$$

Since Ω is a symplectic form, $\sigma_*\xi = 0$. Consequently, if f is a smooth function on M, then

$$\xi \cdot (f \circ \sigma) = i(\xi)\sigma^* \mathrm{d}f = \sigma^* i(\sigma_* \xi)\,\mathrm{d}f = 0.$$

Therefore we can define a unique vector field X_f characterized by

$$i(X_f)\omega = f \circ \sigma \quad \text{and} \quad i(X_f)\,\mathrm{d}\omega = -\mathrm{d}(f \circ \sigma).$$

It is clear from Cartan's formula that $L_{X_f}\omega = 0$. For convenience we put $G = \pi_1(M) \rtimes \mathbb{R}$. We have now to prove that X_f is G-invariant. It suffices to evaluate separately $i(g_* X_f)\omega$ and $i(g_* X_f)\,\mathrm{d}\omega$, for all $g \in G$. We have

$$i(g_* X_f)\omega = g^{-1*}[i(X_f)(g^*\omega)] = g^{-1*}[i(X_f)\omega] = g^{-1*}(f \circ \sigma) = f \circ \sigma \circ g^{-1}$$

since $\sigma \circ g^{-1} = \sigma$, we get

$$i(g_* X_f)\omega = f \circ \sigma.$$

Similar computations lead to

$$i(g_* X_f)\,\mathrm{d}\omega = -\mathrm{d}(f \circ \sigma).$$

Hence $g_* X_f = X_f$ so $X_f \in \mathcal{L}$. Conversely, any $X \in \mathcal{L}$ defines a function $f = i(X)\omega \in C^\infty(Y, \mathbb{R})$. As

$$L_X\omega = i(X)\,\mathrm{d}\omega + \mathrm{d}(i(X)\omega) = 0,$$

we get

$$\xi \cdot f = i(\xi)\,\mathrm{d}f = i(\xi)\,\mathrm{d}(i(X)\omega) = -i(\xi)i(X)\,\mathrm{d}\omega = i(X)i(\xi)\,\mathrm{d}\omega = 0.$$

Hence f is basic for the projection π. Moreover, if $g \in G$ we have

$$\begin{aligned} g^* f &= f \circ g = g^*(i(X)\omega) = i(g_*^{-1}X)g^*\omega \\ &= i(X)\omega = f \quad (\text{since } g_* X = X \text{ and } g^*\omega = \omega). \end{aligned}$$

Finally f is basic for $p \circ \pi$. Obviously $f \mapsto X_f$ is a linear isomorphism from $P(M, \Omega)$ onto $\mathcal{L}$. The verification of the relation

$$X_{\{f,g\}} = [X_f, X_g]$$

is straightforward and reproduces word by word the classical situation in prequantization [17]. This completes the proof of Theorem 3. □

Examples 5.

(1) If (M, Ω) is riemannian with nonpositive curvature, it is a classical fact (Cartan) that $\widetilde{M}$ is diffeomorphic to an euclidean space. Our theorem then applies in this case. This includes all surface of genus bigger than one.

(2) If G is a finite dimensional Lie group; its universal covering $\widehat{G}$ satisfies $H^2(\widehat{G}, \mathbb{R}) = 0$, for the second homotopy group of any finite dimensional Lie group is trivial [6]. Hence a Lie group (G, Ω) equipped with a symplectic form satisfies the hypothesis of our theorem.

Remark 6. If the above action of $\pi_1(M)$ is totally discontinuous, then the orbit space $N = Y/(\pi_1(M))$ is a smooth manifold and the natural projection $\pi : N \to M$ is a S^1 bundle. Moreover the contact form on Y descends to a contact form ω^* on N such that $\pi^*\Omega = d\omega^*$, and we are back in the prequantization framework as in Remark 2. The Lie algebra $\mathcal{L}$ descends to the Lie algebra of strictly contact vector fields of (N, ω^*). We will see (Theorem 9) that this Lie algebra is integrable. In general to check the integrability of $\mathcal{L}$, one should refer to results of [9] or [11]. We have not been able to do it so far.

3. Integrability

Let us first precise the topology on the considered spaces. If (M, Ω) is a symplectic manifold, we denote by $C_c^\infty(M)$, $\mathrm{Ham}_c(M)$, $\mathrm{Diff}_{\Omega,c}(M)$ respectively the space of smooth compactly supported real functions on M, the globally hamiltonian compactly supported vector fields and the group of compactly supported symplectic diffeomorphisms of M. $\mathrm{Diff}_{\Omega,c}(M)$ (and similarly the other spaces introduced above) will be endowed with inductive limit topology:

$$\mathrm{Diff}_{\Omega,c}(M) = \varinjlim_K \mathrm{Diff}_\Omega(M)_K,$$

where K runs over all compact subsets of M and $\mathrm{Diff}_\Omega(M)_K$ is the group of symplectic diffeomorphisms supported in K with the C^∞ topology. We denote by $G_\Omega(M)$ the identity component in $\mathrm{Diff}_{\Omega,c}(M)$ and by $\widetilde{G_\Omega(M)}$ its universal covering.

We now recall the definition of the Calabi homomorphism

$$\widetilde{S} : \widetilde{G_\Omega(M)} \to H_c^1(M, \mathbb{R}).$$

Let

$$I_\Omega(M) = \{\gamma : [0, 1] \to \mathrm{Diff}^\infty_{\Omega,c}(M) / \gamma(0) = \mathrm{id}_M \text{ and } (t, x) \mapsto \gamma(t)(x) \text{ is } C^\infty\}$$

be the set of compactly supported symplectic isotopies, equipped with the compact-open C^1 topology. Then $\widetilde{G_\Omega(M)} = I_\Omega(M)/\!\sim$, where $\gamma \sim \mu$ iff they are homotopic in $\mathrm{Diff}^\infty_{\Omega,c}(M)$ with fixed end points. $\pi_1(G_\Omega(M))$ is then equal to the kernel of the natural projection $\widetilde{G_\Omega(M)} \to G_\Omega(M)$, namely: the homotopy classes of identity-based loops in $G_\Omega(M)$.

If $\gamma \in I_\Omega(M)$ we define a family of vector fields $\dot{\gamma}_t$ in $\mathcal{L}_\Omega(M)_c$ by

$$\dot{\gamma}_t(x) = \frac{d}{ds}\gamma_s(\gamma_t^{-1}(x))\Big|_{s=t}.$$

The mapping, with values in the space $Z_c^1(M)$ of closed compactly supported 1-forms, defined by

$$I_\Omega(M) \ni \gamma \mapsto \int_0^1 \gamma_t^* i(\dot{\gamma}_t)(\Omega)\, dt \in Z_c^1(M)$$

induces a well defined surjective and continuous homomorphism $\widetilde{S}$, which is called the Calabi homomorphism. Denote now by Γ the subgroup of $H_c^1(M, \mathbb{R})$:

$$\Gamma = \widetilde{S}(\pi_1(G_\Omega(M))).$$

Finally we denote by

$$S : G_\Omega(M) \to H_c^1(M, \mathbb{R})/\Gamma,$$

the induced homomorphism. We have the following result.

Theorem 7. *Suppose that Γ is totally disconnected (in particular if Γ is discrete), then:*

(1) *if the symplectic manifold (M, Ω) is noncompact, $P_c(M, \Omega)$ is integrated by the kernel* $\ker S$;

(2) *if the symplectic manifold (M, Ω) is compact, then $P(M, \Omega)$ is integrated by* $[G_\Omega(M), G_\Omega(M)] \times S^1$.

Proof. The main ingredient is the Weinstein chart [19] which we now analyze closely. The graph $\{(x, \phi(x)) \in M \times M\}$ of a symplectic diffeomorphism ϕ of (M, Ω) is a lagrangian submanifold of $(M \times M, \Omega^*)$ where $\Omega^* = \pi_1^*\Omega - \pi_2^*\Omega$ (π_i denoting the projections of $M \times M$ onto each factor. The graph of the identity is the diagonal Δ. If ϕ has support in K then its graph coincides with Δ outside of K. By a theorem of Kostant–Weinstein [20], a neighborhood U of Δ in $M \times M$ is symplectically isomorphic with a neighborhood of the zero section of $T^*\Delta \simeq T^*M$. Hence symplectic diffeomorphisms which are C^0 close to the identity are in one-to-one correspondence with lagrangian submanifolds of T^*M C^0 close to the zero section. If the symplectic diffeomorphism is C^1 close to the identity then it corresponds to a submanifold of T^*M which is the graph of a closed 1-form. This is the Weinstein chart $W : \mathcal{U} \to Z^1(M)$, where $\mathcal{U}$ denotes a neighborhood of the identity in the C^1 topology. Clearly, the supports of ϕ and $W(\phi)$ coincide. We then get a chart $W : \mathcal{U} \to Z_c^1(M)$ from a neighborhood of the identity in $\mathrm{Diff}^\infty_{\Omega,c}(M)$ with a neighborhood of zero in the space of compactly supported closed 1-forms. This means that the group of compactly supported symplectic diffeomorphisms of a noncompact symplectic manifold integrates the Lie algebra of symplectic vector fields with compact supports.

If $h \in \mathcal{U}$ we let $[W(h)] \in H_c^1(M, \mathbb{R})$ be the cohomology class of $W(h)$. Let $h_t^c = W^{-1}(t \cdot W(h))$ be the "canonical" isotopy from h to the identity, it is proved in [3] that

$$\widetilde{S}(\{h_t^c\}) = [W(h)].$$

Therefore if $h \in \ker S$ then $\widetilde{S}(h_t^c) \in \Gamma$. Moreover, by continuity of $\widetilde{S}$, $\widetilde{S}(\{h_t^c\})$ is close to zero in $H_c^1(M, \mathbb{R})$ provided that h is close enough to the identity. Since the connected components of Γ are just single points $[W(h)] = 0$, i.e. $W(h)$ is an exact compactly supported 1-form.

Therefore the group ker S is modeled on the space $B^1_c(M)$ of exact 1-forms on M with compact supports. Note that, by the very definition, $\mathrm{Ham}_c(M)$ is isomorphic to $B^1_c(M)$.

On the one hand, if M is not compact, $B^1_c(M)$ and $C^\infty_c(M)$ are isomorphic. For $h \in \ker S$ therefore there exists a unique function f_h with compact support such that $W(h) = \mathrm{d}(f_h)$. For each point $x \in M$, f_h is defined as

$$f_h(x) = \int_{\gamma_x} W(h),$$

where γ_x is an arbitrary C^∞ path joining x to an arbitrary fixed point x_0. Clearly $h \mapsto f_h$ is a smooth correspondence between a neighborhood $\mathcal{V} = \mathcal{U} \cap \ker S$ of the identity and a neighborhood of zero in $C^\infty_c(M)$. This shows that if M is not compact, then $P_c(M, \Omega)$ is modeled on ker S. It remains to check that the Lie algebra κ of ker S is isomorphic to $P_c(M, \Omega)$. As $P_c(M, \Omega)$ is identified with $\mathrm{Ham}_c(M)$, we shall simply point out that κ consists of $\mathrm{Ham}(M)$ (for compact or noncompact M). In fact, a vector field X belongs to κ if X is the velocity of an isotopy c_t lying in ker S and such that $c_0 = \mathrm{id}_M$ (i.e. $X(x) = (\mathrm{d}/\mathrm{d}t)c_t(x)|t = 0$); it is shown in [3] that $i(\dot{c}_t)\Omega$ is exact at any time t. In particular $X = \dot{c}_0$ satisfies $i(X)\Omega = \mathrm{d}f$, and $X \in \mathrm{Ham}(M)$. Conversely, let X denote a hamiltonian vector field and ϕ_t its flow. We have to verify that $\phi_t \in \ker S$, this is equivalent to the fact that, for all t, there exists an isotopy γ_s joining ϕ_t to id_M and such that $\int_0^1 i(\dot{\gamma}_s)\Omega \,\mathrm{d}s$ is exact. Taking $\gamma_s = \phi_{st}$ we get $\int_0^1 i(\dot{\gamma}_s)\Omega \,\mathrm{d}s = t\,\mathrm{d}f$ where $i(X)\Omega = \mathrm{d}f$, so $\phi_t \in \ker S$ and $X \in \kappa$.

On the other hand, if M is compact, we saw that $C^\infty(M) = \mathrm{Ham}(M) \times \mathbb{R}$ as Lie algebras. Therefore, because $\mathrm{Ham}(M)$ is integrated by ker S, then $C^\infty(M) = \mathrm{Ham}(M) \times \mathbb{R}$ is integrated by $\ker S \times S^1$. But for M compact, a deep theorem of [2] asserts that ker S is equal to the commutator subgroup $[G_\Omega(M), G_\Omega(M)]$. Therefore $[G_\Omega(M), G_\Omega(M)] \times S^1$ integrates $C^\infty(M)$. □

Remarks 8.

(1) The integrability of $P(M, \Omega)$ in the compact case was first observed in [1] by different methods.

(2) The existence of Weinstein's chart was crucial in the proof of Theorem 7. Likewise, Lychagin's generalization of Weinstein chart [13] allows to integrate the Lie algebra of compactly supported contact vector fields of a noncompact contact manifold. The functional analysis approaches do not seem to work in the noncompact case.

Let us now consider the case of a regular contact manifold (N, ω) [5]. This means that the orbits of the Reeb fields ξ of ω are all circles and induce a free S^1 action on N. By Boothby–Wang theorem [5], the orbit space is a smooth symplectic manifold (M, Ω) such that if $\pi: N \to M$ denotes the projection, then $\pi^*\Omega = \mathrm{d}\omega$. Let $G_\omega(N)$ be the identity component in the group of diffeomorphisms of N preserving the contact form ω, with the C^∞ topology. We saw in Remark 2 that $\mathcal{L}_\omega(N)$ is isomorphic to $P(M, \Omega)$ and hence is integrated by $[G_\Omega(M), G_\Omega(M)] \times S^1$. We show next that $\mathcal{L}_\omega(N) \approx P(M, \Omega)$ is integrated by another group.

Theorem 9. *Let (N, ω) be a compact regular contact manifold. Then $G_\omega(N)$ is modeled on $\mathcal{L}_\omega(N)$.*

Proof. Let $\pi : N \to M$ be the Boothby–Wang bundle, over the symplectic manifold (M, Ω).

It is enough to show that $G_\omega(N)$ is locally isomorphic with $\ker S \times S^1$. Indeed, it is proved in [4] that $\mathcal{L}_\omega(N)$ is isomorphic as Lie algebra with $\mathrm{Ham}(M) \times \mathbb{R} \approx B^1(M) \times \mathbb{R}$, and we just proved in Theorem 7 that $\ker S$ integrates $\mathrm{Ham}(M)$. Hence $G_\omega(N)$ is modeled on $\mathrm{Ham}(M) \times \mathbb{R} \approx \mathcal{L}_\omega(N)$.

There is a natural projection $p : G_\omega(N) \to \ker S$, $\ker S \subset G_\Omega(M)$, with kernel S^1. It is enough to construct a local section over a domain $\mathcal{V} \subset \ker S$ of the chart $W : \mathcal{V} \to B^1(M)$ defined in Theorem 7. For $h \in \mathcal{V}$, consider the isotopy: $h_t = W^{-1}(t \cdot W(h)) \in \ker S$ and the family of vector fields $\dot{g}_t$ defined by the isotopy g_t earlier in this section. By the Hodge–de Rham theorem, there is a smooth family of functions f_t such that: $i(\dot{g}_t)\Omega = \mathrm{d}f_t$. Recall that the contact form ω is a connection on the Boothby–Wang bundle. Now for each t, let X_t be the horizontal lift of $\dot{g}_t$ (via the connection ω). Following [4], we consider the family of vector fields

$$Y_t = X_t - (f_t \circ \pi)\xi.$$

One shows easily that $L_{Y_t}\omega = 0$, and that the family $\phi_t \in G_\omega(N)$ of diffeomorphisms of N obtained by integrating the family of vector fields Y_t covers g_t. The section σ over $\mathcal{V}$ is then defined by $\sigma(h) = \phi_1$. □

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